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Neighborhood unions and regularity in graphs

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Abstract

One way to generalize the concept of degree in a graph is to consider the neighborhood $N(S)$ of an independent set S instead of a simple vertex. The minimum generalized degree of order t of G is then defined, for $1 \leq t \leq \alpha$ (the independence number of G), by $u_t = \min\{|N(S)| : S \subset V, S \text{ is independent and } |S| = t\}$. The graph G is said to be u_t -regular if $|N(S_1)| = |N(S_2)|$ for every pair S_1, S_2 of independent sets of t elements, totally u_t -regular (resp. totally $u_{t \leq s}$ -regular where s is given $\leq \alpha$) if it is u_t -regular for every $t \leq \alpha$ (resp. for every $t \leq s$), strongly u_t -regular (resp. strongly $u_{t \leq s}$ -regular) if $|N(S_1)| = |N(S_2)|$ for every pair S_1, S_2 of independent sets of G (resp. every pair of independent sets of order at most s). We determine the strongly $u_{t \leq 2}$ -regular graphs and give some properties of the totally $u_{t \leq 2}$ -regular and totally u_t -regular graphs. Some of our results improve already known results. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Graph; Regular; Neighborhood union

1. Introduction

In a simple graph $G = \langle V, E \rangle$ of order $|V| = n$, the neighborhood of a vertex x is $N(x) = \{y \in V : xy \in E\}$ and its degree is $d(x) = |N(x)|$. The minimum (resp. maximum) degree of G is $\delta(G) = \min\{d(x) : x \in V\}$ (resp. $\Delta(G) = \max\{d(x) : x \in V\}$). The graph G is regular if $d(x_1) = d(x_2)$ for every pair of vertices of G , in other terms if $\delta(G) = \Delta(G)$.

For several years, many authors proposed various generalizations of the concept of degree. In the generalization which is considered here, the neighborhood of a vertex is replaced by the neighborhood $N(S) = \bigcup_{x \in S} N(x)$ of an independent set S . Hence for every positive integer $t \leq \alpha$, the independence number (that is, the maximum number of pairwise nonadjacent vertices) of G , we define $u_t(G) = \min\{|N(S)| : S \subset V, S \text{ is independent and } |S| = t\}$ and $U_t(G) = \max\{|N(S)| : S \subset V, S \text{ is independent and } |S| = t\}$.

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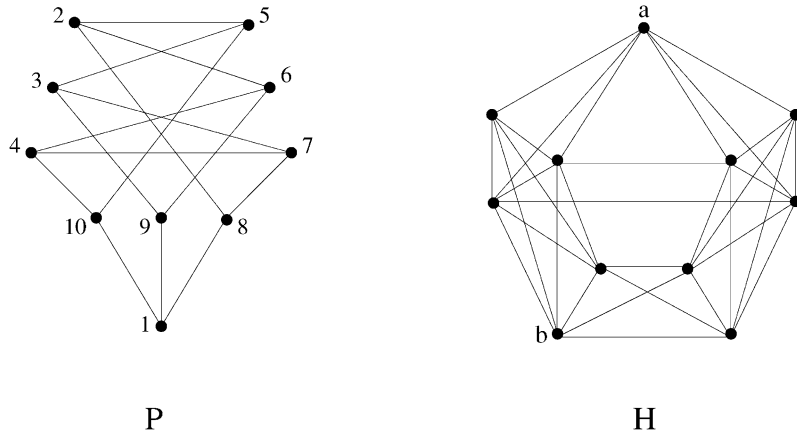


Fig. 1.

When there is no ambiguity, we write δ, Δ, u_t, U_t for $\delta(G), \Delta(G), u_t(G), U_t(G)$, respectively. Clearly $u_1 = \delta, U_1 = \Delta$ and $u_\alpha = U_\alpha = n - \alpha$.

A graph G is said to be u_t -regular if $|N(S_1)| = |N(S_2)|$ for every pair of independent sets of t elements of G , in other terms if $u_t = U_t$. Note that u_1 -regular means regular and that every graph of independence number α is u_α -regular. A graph G is totally u_t -regular if it is u_t -regular for every t between 1 and α , and totally $u_{t \leq s}$ -regular, where the positive integer s is given $\leq \alpha$, if it is u_t -regular for every t between 1 and s . The graph is strongly u_t -regular if $|N(S_1)| = |N(S_2)|$ for every pair S_1, S_2 of independent sets, and strongly $u_{t \leq s}$ -regular, where s is given $\leq \alpha$, if $|N(S_1)| = |N(S_2)|$ for every pair S_1, S_2 of independent sets of at most s elements of G .

For instance, the Petersen graph P shown in Fig. 1 is u_1 -regular of degree 3 and u_2 -regular with $u_2 = 5$ (it is easy to check that $|N(u, v)| = 5$ for every pair of nonadjacent vertices). But P is not u_3 -regular since $|N(2, 3, 4)| = 6$ and $|N(8, 9, 10)| = 7$. Hence P is totally $u_{t \leq 2}$ -regular but not totally u_t -regular and not strongly $u_{t \leq 2}$ -regular.

The graph H of Fig. 1 is u_1 -regular with $u_1 = 6$, u_2 -regular with $u_2 = 9$, and thus totally u_t -regular since its independence number is $\alpha = 2$, but not strongly u_t -regular.

The complete balanced multipartite graph $K_{p,p,\dots,p}$ is strongly u_t -regular since $|N(S)| = n - p$ for every independent set S .

From the definition of totally u_t -regular graphs, it is natural to look for some relationships between the properties of total u_t -regularity, vertex-transitivity (see e.g. [1]) and strong regularity (a regular graph of diameter 2 is strongly regular if $|N(x) \cap N(y)|$ is equal to a constant λ for any pair of adjacent vertices x and y , and to a constant μ for any pair of nonadjacent vertices x and y , see e.g. [2]).

Clearly, a noncomplete strongly regular graph is totally $u_{t \leq 2}$ -regular. However, the Petersen graph is vertex-transitive and strongly regular, but not totally u_t -regular. Hence, neither the vertex-transitivity nor the strong regularity implies the total u_t -regularity.

The graph H of Fig. 1 is totally u_t -regular but not vertex-transitive since the neighborhoods of a and b are not isomorphic, nor strongly regular since the neighborhood of b is not regular. Hence, at least for $\alpha=2$, the total u_t -regularity does not imply the vertex-transitivity nor the strong regularity.

Recently, some authors initialized the study of the u_t -regularity of a graph, in particular, Haynes and Knisley [4] and Faudree and Knisley [3]. We now cite some of their results which are of interest in this paper.

Theorem 1.1 (Haynes and Knisley [4]). *The diameter of a connected totally u_t -regular graph is at most 2.*

Theorem 1.2 (Haynes and Knisley [4]). *A graph G is strongly u_t -regular if and only if G is a balanced complete multipartite graph $K_{\alpha, \alpha, \dots, \alpha}$ with $l = n/\alpha \geq 1$ parts.*

In [3], the authors suppose that the value of the generalized minimum degree u_2 is fixed and that n can grow to infinity. Under this condition, they describe all u_2 -regular graphs of large order. None of these graphs is regular, which shows that for u_2 fixed and sufficiently large n , there do not exist totally $u_{t \leq 2}$ -regular graphs.

In this paper we improve Theorems 1.1 and 1.2 by showing that the conclusions remain true with weaker hypotheses (cf. Theorems 2.1 and 3.2), and we give some properties of totally u_t -regular graphs.

2. Totally u_t -regular graphs

Our first result deals with the diameter of connected graphs and can be compared with Theorem 1.1.

Theorem 2.1. *The diameter of a connected totally $u_{t \leq 2}$ -regular graph is equal to 2.*

Proof. We suppose $\alpha \geq 2$ and thus G is not complete. If $\text{diam}(G) \geq 3$, let u and v be two vertices at distance 3 and $u x_1 x_2 v$ an induced path of G . Since the two nonadjacent vertices u and v have no common neighbors, $|N(u, v)| = |N(u)| + |N(v)| = 2\delta$. Since the two nonadjacent vertices u and x_2 have at least one common neighbor, $|N(u, v)| \leq |N(u)| + |N(v)| - 1 = 2\delta - 1$. We get a contradiction with the hypothesis of u_2 -regularity of G . Hence $\text{diam}(G) = 2$. \square

Note that the two possible converses, namely if $\text{diam}(G) = 2$ and G is regular then it is u_2 -regular, or if $\text{diam}(G) = 2$ and G is u_2 -regular then it is regular, are false. For instance, the graph G consisting of a clique $A \simeq K_l$, an independent set $B \simeq \bar{K}_r$ with $r \geq 2$, and all the edges between A and B is u_2 -regular with $u_2 = l$, has diameter 2 but is not regular. The graph R of Fig. 2 is regular of degree 4, has diameter 2 but is not u_2 -regular since $|N(z, t)| = 5$ and $|N(x, y)| = 6$.

The second result deals with well-covered graphs, that is graphs such that every maximal independent set is maximum. The reader can find a survey of well-covered graphs in [5].

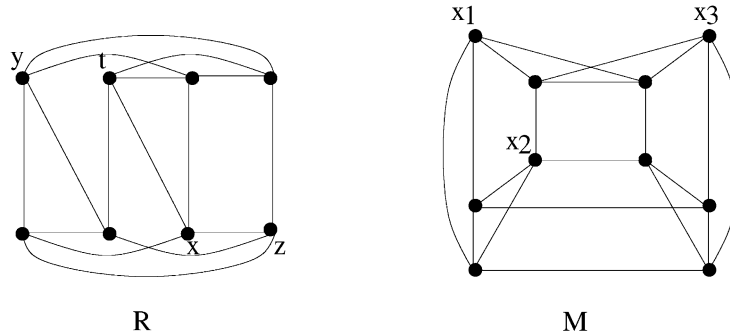


Fig. 2.

Lemma 2.2. *If the graph G is totally $u_{t \leq s}$ -regular for some $s < \alpha$, then every maximal independent set of G has more than s elements.*

Proof. Suppose G contains a maximal independent set S of $q \leq s$ elements; then $|N(S)| = n - q$. Let S' be a maximum independent set of G and S'' a subset of q elements of S' . Since $q \leq s$ and G is u_q -regular, $|N(S'')| = |N(S)| = n - q \geq n - s > n - \alpha = |N(S')|$, in contradiction to $S'' \subset S'$. \square

Theorem 2.3. *Every totally u_t -regular graph is well-covered.*

Proof. If G is totally u_t -regular then by Lemma 2.2, every maximal independent set has more than s elements for all $s < \alpha$. Hence every maximal independent set has α elements and thus G is well-covered. \square

Corollary 2.4. *Every connected bipartite totally u_t -regular graph is a balanced complete bipartite graph.*

Proof. This is an obvious consequence of a result of Ravindra [6] saying that every connected, bipartite, regular and well-covered graph is complete bipartite. \square

In view of Theorems 2.1 and 2.3, we can wonder if a connected well-covered graph G of diameter 2 is necessarily u_t -regular or $u_{t \leq 2}$ -regular. A negative answer is given by the graph M of Fig. 2 which is regular of degree 4, well-covered (with $\alpha = 3$), of diameter 2, but not $u_{t \leq 2}$ -regular since $|N(x_1, x_2)| = 5$, $|N(x_1, x_3)| = 6$ and $|N(x_2, x_3)| = 7$.

We give now some structural properties of totally u_t -regular graphs.

If v is a vertex of an independent set S , we call S -private neighborhood of v , denoted by $I(v, S)$, the set $N(S) \setminus N(S \setminus \{v\})$ of the vertices which are adjacent to v and to no other vertex of S .

Lemma 2.5. *If $S = \{v_1, v_2, \dots, v_k\}$ is an independent set of a graph G , then $N(S)$ is the disjoint union $I(v_1, S) \cup I(v_2, S \setminus \{v_1\}) \cup I(v_3, S \setminus \{v_1, v_2\}) \cup \dots \cup I(v_{k-1}, S \setminus \{v_1, v_2, \dots, v_{k-2}\}) \cup N(v_k)$.*

The proof by induction on k is obvious and we omit it. Note that for the sake of symmetry, we can write $N(v_k) = I(v_k, S \setminus \{v_1, v_2, \dots, v_{k-1}\})$.

Consider again the example of the Petersen graph P of Fig. 1 which is u_1 -regular and u_2 -regular but not u_3 -regular. Let S be any independent set and v any vertex of S . If $|S| = 1$ then $|I(v, S)| = \delta = 3$ and one can check that if $|S| = 2$, then $|I(v, S)| = 2$. But there exist independent sets S of order 3 and vertices v of S such that the values of $|I(v, S)|$ are different. For instance, if $S = \{1, 2, 7\}$ then $|I(1, S)| = 2$ and if $S = \{1, 2, 3\}$ then $|I(1, S)| = 1$.

Similarly in the graph M of Fig. 2 which is regular but not $u_{t \leq 2}$ -regular, if $S = \{x_1, x_2\}$ then $|I(x_1, S)| = 1$, if $S = \{x_1, x_3\}$ then $|I(x_1, S)| = 2$ and if $S = \{x_2, x_3\}$ then $|I(x_2, S)| = 3$.

These observations lead to the following definition.

Definition 2.6. A graph G has the property (\mathcal{P}_s) if for every pair S, S' of independent sets of $q \leq s$ elements of G and for every pair of vertices $v \in S$ and $v' \in S'$, $|I(v, S)| = |I(v', S')|$.

Note that if G satisfies (\mathcal{P}_s) , it satisfies $(\mathcal{P}_{s'})$ for all $s' \leq s$.

Theorem 2.7. A graph G is totally $u_{t \leq s}$ -regular if and only if it satisfies (\mathcal{P}_s) .

Proof. 1. Suppose G totally $u_{t \leq s}$ -regular and let S, S' be any two independent sets of $q \leq s$ elements of G , v, v' any two vertices, respectively, belonging to S and S' . If $|S| = |S'| = 1$, then obviously $|I(v, S)| = |I(v', S')| = \delta$. For $q \geq 2$, $|N(S)| = |N(S \setminus \{v\})| + |I(v, S)|$ and $|N(S')| = |N(S' \setminus \{v'\})| + |I(v', S')|$. Since G is u_q -regular and u_{q-1} -regular, $|N(S)| = |N(S')|$ and $|N(S \setminus \{v\})| = |N(S' \setminus \{v'\})|$. Therefore $|I(v, S)| = |I(v', S')|$ and G satisfies (\mathcal{P}_s) .

2. The converse part is shown by induction on s . If G satisfies (\mathcal{P}_1) then for any vertices v and v' , $d(v) = d(v')$ and G is u_1 -regular. Suppose that if G satisfies (\mathcal{P}_{s-1}) with $2 \leq s \leq \alpha$ then G is totally $u_{t \leq s-1}$ -regular, and let G be a graph satisfying (\mathcal{P}_s) . Since (\mathcal{P}_s) implies (\mathcal{P}_{s-1}) , G is $u_{t \leq s-1}$ -regular by the induction hypothesis. Let S and S' be two independent sets of s elements of G , and $v \in S$, $v' \in S'$. Then $|N(S)| = |I(v, S)| + |N(S \setminus \{v\})|$ and $|N(S')| = |I(v', S')| + |N(S' \setminus \{v'\})|$. But $|I(v, S)| = |I(v', S')|$ since G satisfies (\mathcal{P}_s) , and $|N(S \setminus \{v\})| = |N(S' \setminus \{v'\})|$ since G is u_{s-1} -regular. Hence $|N(S)| = |N(S')|$ and thus G is also u_s -regular, which achieves the proof. \square

For a totally u_t -regular graph G , we denote by $n_i(G)$, or n_i for short, the common value of $|I(v, S)|$ for all independent sets S of i elements of G and all vertices $v \in S$. Some properties of the sequence $(n_i)_{1 \leq i \leq \alpha}$ are given below.

Theorem 2.8. The sequence $n_1, n_2, \dots, n_\alpha$ of nonnegative integers which is associated to a totally u_t -regular graph G satisfies the following properties:

1. $n_1 = \delta$;
2. For every $1 \leq t \leq \alpha$, $u_t = \sum_{i=1}^t n_i$ and in particular $n - \alpha = \sum_{i=1}^\alpha n_i$;

3. The sequence n_i is nonincreasing;
4. $n_2 < \delta$ strictly.

Proof. 1. If $|S| = 1$ then $|I(v, \{v\})| = d(v) = \delta$.

2. This is a consequence of Lemma 2.5.

3. Let S be an independent set of $i \geq 2$ elements, u, v two vertices of S , and let $Y = (N(u) \cap N(v)) \setminus N(S \setminus \{u, v\})$. The set $I(u, S \setminus \{v\})$ is the disjoint union $I(u, S) \cup Y$. By the definition of the n_i 's, $|I(u, S \setminus \{v\})| = n_{i-1}$ and $|I(u, S)| = n_i$. Hence $n_{i-1} \geq n_i$.

4. Let $S = \{u, v\}$ be an independent set of G . By Theorem 2.1, u and v have at least one common neighbor. Hence $|I(u, S)| < d(u) = \delta$ and thus $n_2 < n_1$. However, one can say nothing about the other terms of the sequence. For instance, for $G \simeq K_{p,p}$, $n_1 = p$, $n_2 = n_3 = \dots = n_{\alpha=p} = 0$. \square

To complete this section, we remark that if α is bounded above by α_0 , then to determine if a given graph G of independence number α is totally u_t -regular is polynomial in n since we have only a polynomial number $\sum_{p=1}^{\alpha_0} \binom{n}{p}$ of subsets S to examine (test the independence and possibly determine $|N(S)|$). One can pose the following question: is this problem still polynomial when nothing is known about α ?

3. Strongly u_t -regular graphs

In this section we improve Theorem 1.2, in a similar way as for Theorem 1.1, by replacing the hypothesis “strongly u_t -regular” by the weaker one “strongly $u_{t \leq 2}$ -regular”.

Lemma 3.1. *If G is strongly $u_{t \leq 2}$ -regular then every pair x_1, x_2 of nonadjacent vertices satisfies $N(x_1) = N(x_2)$.*

Proof. In every graph G , any nonadjacent vertices x_1 and x_2 satisfy $|N(x_1, x_2)| = |N(x_1)| + |N(x_2)| - |N(x_1) \cap N(x_2)|$. If G is strongly $u_{t \leq 2}$ -regular then $|N(x_1)| = |N(x_2)| = |N(x_1, x_2)| = \delta$. Hence, $|N(x_1) \cap N(x_2)| = |N(x_1)| = |N(x_2)|$ and thus $N(x_1) \cap N(x_2) = N(x_1) = N(x_2)$. \square

Theorem 3.2. *Every strongly $u_{t \leq 2}$ -regular graph is a complete balanced multipartite graph $K_{\alpha, \alpha, \dots, \alpha}$ with $\alpha \geq 2$ and where the number $l = n/\alpha$ of parts is at least 1.*

Proof. We proceed by induction on the order n of G which is at least 2 since $\alpha(G) \geq 2$. If $n = 2$ then $\alpha(G) = 2$ and G is the independent set \bar{K}_2 which is one-partite. Suppose the property true for a graph of order less than n and let G be a strongly $u_{t \leq 2}$ -regular graph of order $n(G) = n$. Let x_1 and x_2 be two nonadjacent vertices of G . By Lemma 3.1, $N(x_1) = N(x_2)$. The vertices x_i different from x_1 and x_2 of the set $S_1 = V \setminus N(x_1)$ are not adjacent to x_1 and thus, again by Lemma 3.1, have all $N(x_1)$ as their neighborhood. Hence the set S_1 is independent. Consider now the graph G_1

induced by $N(x_1)$ in G . Its order is $n(G_1) = n(G) - |S_1|$. The neighborhood in G of every vertex y of $N(x_1)$ contains S_1 , and its neighborhood in G_1 is $N_{G_1}(y) = N_G(y) \setminus S_1$. Therefore G_1 is regular of degree $\delta(G_1) = \delta(G) - |S_1|$. Since $\alpha(G) \geq 2$, $\delta(G) < n(G) - 1$ and thus $\delta(G_1) < n(G) - |S_1| - 1 = n(G_1) - 1$. Hence $\alpha(G_1) \geq 2$. If y and z are two nonadjacent vertices of G_1 , then $|N_{G_1}(y) \cup N_{G_1}(z)| = |(N_G(y) \cup N_G(z)) \setminus S_1| = u_2(G) - |S_1|$. Therefore G_1 is u_2 -regular with $u_2(G_1) = u_2(G) - |S_1|$. Since $\delta(G) = u_2(G)$, we get $\delta(G_1) = u_2(G_1)$ and thus G_1 is strongly $u_{t \leq 2}$ -regular. By the induction hypothesis, G_1 is a graph $K_{\alpha_1, \alpha_1, \dots, \alpha_1}$ where $\alpha_1 = \alpha(G_1)$ and where the number l_1 of parts of this complete balanced multipartite graph satisfies $\alpha_1 l_1 = n_1(G) = n(G) - |S_1|$. The degrees in G of a vertex of $N(x_1)$ and of a vertex of S_1 are, respectively $(l_1 - 1)\alpha_1 + |S_1| = n(G) - \alpha_1$ and $n(G) - |S_1|$. Since G is regular, $|S_1| = \alpha_1$ and G is the complete balanced multipartite graph $K_{\alpha_1, \alpha_1, \dots, \alpha_1}$ with $l_1 + 1 = n(G)/\alpha_1$ parts, which achieves the proof. \square

The following corollary is now obvious.

Corollary 3.3. *Every strongly $u_{t \leq 2}$ -regular graph is strongly u_t -regular.*

Remark. (1) Theorem 3.2 does not contradict the result of Faudree and Knisley cited in the introduction, which said that for u_2 fixed and n sufficiently large, there do not exist totally $u_{t \leq 2}$ -regular graphs, and a fortiori strongly u_t -regular graphs, of order n . Indeed the graphs $K_{\alpha, \alpha, \dots, \alpha}$ obtained in Theorem 3.2 have $\delta = u_2 = n - \alpha \geq n/2$ and thus δ has same order of magnitude as n .

(2) Corollary 3.3 has no exact counterpart for totally u_t -regular graphs. In other terms, a totally $u_{t \leq 2}$ -regular graph is not necessarily totally u_t -regular as shown by the Petersen graph. Another example with $u_t \leq 3$ instead of $u_t \leq 2$ is given by the graph L of Fig. 3 which is 5-regular of order 16 and independence number $\alpha = 5$. This graph

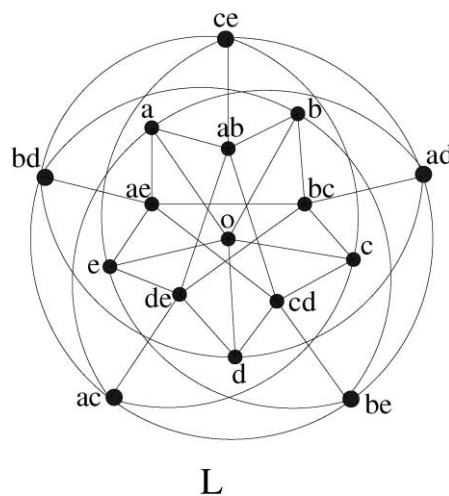


Fig. 3.

is called the Greenwood-Gleason graph in [1] p. 242, and is the complement of the Clebsch graph described in [2, p. 104]. The graph L is u_2 -regular with $u_2 = 8$, u_3 -regular with $u_3 = 10$, but not u_4 -regular since $|N(\{a, b, c, d\})| = 11$ and $|N(\{o, ab, bc, ac\})| = 12$. Hence L is totally $u_{t \leq 3}$ -regular and not totally u_t -regular.

We pose a second question:

Does there exist a positive integer t_0 such that every totally $u_{t \leq t_0}$ -regular graph is totally u_t -regular?

Clearly, if the answer to this second question is affirmative, our first question, relative to the complexity of the recognition of totally u_t -regular graphs, has no more interest.

Acknowledgements

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